ON A CERTAIN PROPERTY OF SUPERSONIC GAS FLOWS

(OB ODNOM SVOISTVE TRANSZVUKOVYKH TECHENII GAZA)

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When investigating gas flows in Laval nozzles great difficulties are encountered in the construction of the flow in the neighborhood of the throat of the channel, where transition from subsonic to supersonic velocities takes place. In this region the gas motion is described by equations of mixed elliptic-hyperbolic type, the general properties of which have not been studied sufficiently up to the present. As is known, the calculation of the supersonic part of the flow is substantially simplified in the particular case in which the sonic surface is a plane perpendicular to the streamlines crossing it. In this case we can investigate separately the subsonic flow region, which is described by equations of elliptic type, and the supersonic, in which the flow is described by equations of hyperbolic type. The sonic plane serves in this case also as a characteristic surface which divides the two regions of gas motion. A great number of papers [1-7] have been devoted to the investigation of supersonic flows with a plane surface of transition through the speed of sound.

In this paper we shall derive the general conditions for which the surface of transition from subsonic to supersonic velocities coincides with a characteristic surface of the equations of gasdynamics. The case under consideration is the only one for which the supersonic flow field may be calculated independently of its subsonic part, because the transonic mixed system of partial differential equations breaks down into purely hyperbolic and purely elliptic parts with the conditions prevailing on the surface of transition known.

The system of equations of gasdynamics may be written in the form

$$v_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = 0, \qquad \frac{\partial \rho v_j}{\partial x_j} = 0, \qquad v_j \frac{\partial s}{\partial x_j} = 0 \qquad (1)$$

$$p = p(\rho, s)$$

where v_i, p, ρ and s denote the velocity components of the flow, pressure, density and entropy at the point with Cartesian coordinates x_i. The usual tensor notation for summation over the repeating indexes i, j is used, where i, j assume the values 1, 2, 3.

The equations which determine the $C_\pm$ characteristic surfaces $x_3 = x_3(x_1, x_2)$ of the system of equations (1), may be written as follows:

$$v_j n_j \pm a = 0 \tag{2}$$

where $a = \sqrt{(\partial p/\partial \rho)_s}$ is the velocity of sound and n_j are the components of the normals to these surfaces, for which the following formulas are valid:

$$n_1 = \frac{1}{k}\frac{\partial x_3}{\partial x_1}, \qquad n_2 = \frac{1}{k}\frac{\partial x_3}{\partial x_2}, \qquad n_3 = -\frac{1}{k} \tag{3}$$

$$k = \sqrt{1 + \left(\frac{\partial x_3}{\partial x_1}\right)^2 + \left(\frac{\partial x_3}{\partial x_2}\right)^2}$$

The system of equations of gasdynamics (1), reduced to the $C_\pm$ characteristics, will assume the form

$$(v_i \pm a n_i)\frac{\partial p}{\partial x_i} + a\rho\,(a\delta_{ij} \pm n_i v_j)\frac{\partial v_i}{\partial x_j} = 0 \tag{4}$$

$$(\delta_{ij} = 1 \text{ for } i = j \text{ and } \delta_{ij} = 0 \text{ for } i \neq j)$$

Equations (4) contain derivatives of the desired functions only along the corresponding characteristic surfaces.

If the $C_\pm$ characteristic surface coincides with the sonic surface, then from Formula (2) there follows that it is orthogonal at any point to the intersecting streamline; therefore, for the vector component of the velocity along this surface we have

$$v_j = \mp a n_j \tag{5}$$

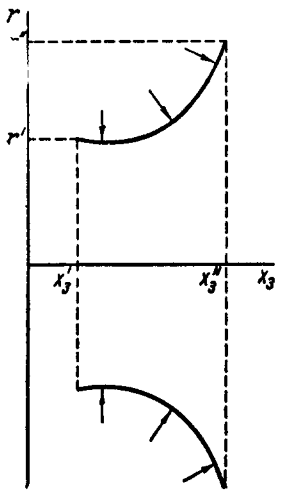
Using Equation (5), we obtain from Equations (4)

$$(\delta_{ij} - n_i n_j)\frac{\partial a n_i}{\partial x_j} = 0$$

Taking into account relations (3) we have finally

$$\frac{\partial}{\partial x_1}\frac{\partial x_3/\partial x_1}{\sqrt{1 + (\partial x_3/\partial x_1)^2 + (\partial x_3/\partial x_2)^2}} + \frac{\partial}{\partial x_2}\frac{\partial x_3/\partial x_2}{\sqrt{1 + (\partial x_3/\partial x_1)^2 + (\partial x_3/\partial x_2)^2}} = 0 \tag{6}$$

Equation (6) is the equation of minimum surfaces. It expresses the fact that at any point the average curvature of the indicated surfaces must be zero. The theory of Equation (6) is closely connected with the theory of analytic functions of a complex variable and is well investigated in [8]. Note that this equation was derived without making any assumptions whatever about the form of the equation of state of the gas or the irrotationality of the flow. The results obtained may be formulated in the form of the following theorem.



Theorem. Let there be an arbitrary closed contour, any point of which is also a point of the surface of transition through the velocity of sound. Let this surface be at the same time a characteristic surface of the equations of gasdynamics. Then this surface will have a minimum area among all the surfaces which may be stretched over the given contour and, furthermore, the velocity vector will be orthogonal to it at each and every point.

If the contour is formed by a plane curve, then the sonic surface passing through it will also be plane, and the tangents to the streamlines at the points of intersection with it will be parallel to each other [1-7].

The sonic surface need not be bounded by a single contour. Cases where it is not are frequently encountered in applied problems. As an example of the case when two closed curves serve as the boundary of the surface of transition, we shall consider the outflow of a gas with sound velocity from a ring-shaped opening located in the peripheral part of an axisymmetrical nozzle (see Figure). Here the surface of transition is confined between two circles located in planes perpendicular to the axis of the channel which we shall make coincident with axis x_3. Let the first circle be situated at a distance x_3' from the origin of coordinates and have a radius r', the second at a distance x_3'' and have a radius r''. The equation of a minimum surface, passing through both circles, will have a form $x_3 = x_3(r)$, where $r = \sqrt{(x_1^2 + x_2^2)}$. Therefore, from relations (6) it follows that

$$r\frac{d^2x_3}{dr^2} + \left(\frac{dx_3}{dr}\right)^3 + \frac{dx_3}{dr} = 0$$

Interchanging dependent and independent variables in this equation, we have

$$r\frac{d^2r}{dx_3^2} = 1 + \left(\frac{dr}{dx_3}\right)^2 \tag{7}$$

The general solution of equation (7) may be represented in the form

$$r = c_1 \cosh\frac{x_3 - c_2}{c_1} \tag{8}$$

where c_1 and c_2 are arbitrary constants. They are determined by the relations

$$r' = c_1 \cosh\frac{x_3' - c_2}{c_1}, \qquad r'' = c_1 \cosh\frac{x_3'' - c_2}{c_1} \tag{9}$$

As seen from Equation (8), a sonic surface in this case is formed by the rotation of a catenary around the axis of the nozzle. It is important to note, however, that in this problem not necessarily one and only one extremum (8) passes through the two given points $A'(x_3', r')$ and $A''(x_3'', r'')$. As the solution of the system of equations (9) shows, depending on the relative location of these points, there may be two, one or no such extrema. In the case when the extrema (8) may not be drawn through the points A' and A'', the discs perpendicular to the axis of the nozzle and located at a distance x_3' and x_3'' from the origin of a coordinate system serve as minimum surfaces.

It is easily proved that in the analogous problem of the outflow of a gas from openings located symmetrically relative to the axis of a channel in the peripheral part of a plane-parallel nozzle, sonic surfaces will be planes connecting the edges of the openings. The solution of such a problem always exists and is unique.

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